Exercise 7

Solve the differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\cos x$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2y_c}{dx^2} - 2\frac{dy_c}{dx} + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad \frac{dy_c}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y_c}{dx^2} = r^2e^{rx}$$

Substitute these formulas into the ODE.

$$r^2e^{rx} - 2(re^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 1 = 0$$

Solve for r.

$$(r-1)^2 = 0$$

$$r = \{1\}$$

Two solutions to the ODE are e^x and xe^x. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$y_c(x) = C_1e^x + C_2xe^x$$

C_1 and C_2 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2y_p}{dx^2} - 2\frac{dy_p}{dx} + y_p = x\cos x \tag{3}$$

Since the inhomogeneous term is a polynomial of degree 1 times cosine, the particular solution is $y_p = (Ax+B)(C\cos x + D\sin x)$.

$$y_p = (Ax+B)(C\cos x + D\sin x)$$

$$\frac{dy_p}{dx} = (A)(C\cos x + D\sin x) + (Ax+B)(-C\sin x + D\cos x)$$

$$\frac{d^2y_p}{dx^2} = (A)(-C\sin x + D\cos x) + (A)(-C\sin x + D\cos x) + (Ax+B)(-C\cos x - D\sin x)$$

Substitute these formulas into equation (3).

$$\begin{aligned}[(A)(-C\sin x + D\cos x) + (A)(-C\sin x + D\cos x) + (Ax+B)(-C\cos x - D\sin x)]\\ -2[(A)(C\cos x + D\sin x) + (Ax+B)(-C\sin x + D\cos x)]\\ +[(Ax+B)(C\cos x + D\sin x)] = x\cos x\end{aligned}$$

Simplify the left side.

$$\begin{aligned}(-2AC + 2AD - 2BD)\cos x + (-2AC + 2BC - 2AD)\sin x\\ + (-2AD)x\cos x + (2AC)x\sin x = x\cos x\end{aligned}$$

Match the coefficients to get a system of equations for A, B, C, and D.

$$\begin{aligned}-2AC + 2AD - 2BD &= 0\\ -2AC + 2BC - 2AD &= 0\\ -2AD &= 1\\ 2AC &= 0\end{aligned}$$

Solving it yields

$$AC = 0 \quad \text{and} \quad AD = -\frac{1}{2} \quad \text{and} \quad BC = -\frac{1}{2} \quad \text{and} \quad BD = -\frac{1}{2}.$$

The particular solution is then

$$\begin{aligned}y_p &= (Ax+B)(C\cos x + D\sin x)\\ &= ACx\cos x + ADx\sin x + BC\cos x + BD\sin x\\ &= -\frac{1}{2}x\sin x - \frac{1}{2}\cos x - \frac{1}{2}\sin x.\end{aligned}$$

Therefore, the general solution to the original ODE is

$$\begin{aligned}y &= y_c + y_p\\ &= C_1e^x + C_2xe^x - \frac{1}{2}x\sin x - \frac{1}{2}\cos x - \frac{1}{2}\sin x.\end{aligned}$$